On Implicitly constituted incompressible fluids

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Formulation of the problem

PROBLEM

$$\left.\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S} &= -\nabla p + \mathbf{b} \\ \mathbb{G}(\mathbb{S}, \mathbb{D}) &= \mathbb{O} \end{aligned}\right\} \text{ in } Q_T$$

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= 0 \\ \mathbf{g}((\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}, \mathbf{v}_{\boldsymbol{\tau}}) &= \mathbf{0} \end{aligned}\right\} \text{ on } \Sigma_T$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \qquad \text{in } \Omega$$

DATA

- $\Omega \subset \mathbb{R}^3$ bounded, open set with $\partial\Omega \in \mathcal{C}^1$ and $\mathbf{n} : \partial\Omega \to \mathbb{R}^3$
- $T > 0$ and $Q_T := (0, T) \times \Omega$, $\Sigma_T := (0, T) \times \partial\Omega$
- $\mathbf{v}_0$, $\mathbf{b}$
- $\mathbb{G}$ and $\mathbf{q}$ - constitutive functions in the bulk and on the boundary

Main questions addressed

FIRST ORDER SYSTEM for the unknown triplet $(\mathbf{v}, p, \mathbb{S})$

$$\left.\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S} &= -\nabla p + \mathbf{b} \\ \mathbb{G}(\mathbb{S}, \mathbb{D}) &= \mathbb{O} \end{aligned}\right\} \text{ in } Q_T$$

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= 0 \\ \mathbf{g}((\mathbb{S}\mathbf{n})_\tau, \mathbf{v}_\tau) &= \mathbf{0} \end{aligned}\right\} \text{ on } \Sigma_T$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \qquad \text{in } \Omega$$

AIM

- To establish large data existence of solution for any set of data $(\Omega,\ T,\ \mathbf{v}_0,\ \mathbf{b})$ and for robust class of constitutive equations described by $\mathbb{G}$ and $\mathbf{g}$
- To develop theory with $p \in L^1(Q_T)$ - important
  - heat-conduting incompressible fluids
  - one/two equation turbulence model
  - incompressible fluids with pressure and shear-rate dependent viscosity
  - corresponding numerical methods and their analysis

Basic information

A PRIORI ESTIMATES

Multiplying the 2nd Eq. by $\mathbf{v}$ ($\mathbf{b} \equiv 0$)

$$\frac{1}{2}\frac{\partial|\mathbf{v}|^2}{\partial t} + \mathrm{div}(\tfrac{1}{2}|\mathbf{v}|^2\mathbf{v}) - \mathrm{div}(\mathbb{S}\mathbf{v}) + \mathbb{S}\cdot\mathbb{D} = -\mathrm{div}(p\mathbf{v})$$

Since $\mathbf{v}\cdot\mathbf{n} = 0$, integrating it over Ω leads to

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_2^2 + \int_\Omega \mathbb{S}\cdot\mathbb{D}\,dx + \int_{\partial\Omega}(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}\cdot\mathbf{v}_{\boldsymbol{\tau}}\,dS = 0$$

The simplest relations

$$\begin{aligned} \mathbb{S} &= 2\nu_*\mathbb{D} \quad \text{in } Q_T \qquad && \nu_* > 0 \\ (\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}} &= \alpha_*\mathbf{v}_{\boldsymbol{\tau}} \quad \text{on } \Sigma_T \qquad && \alpha_* > 0 \end{aligned}$$

$(\mathbf{z})_{\boldsymbol{\tau}} := \mathbf{z} - (\mathbf{z}\cdot\mathbf{n})\mathbf{n}$ $\qquad$ $2\mathbb{D} = 2\mathbb{D}(\mathbf{v}) := \nabla\mathbf{v} + (\nabla\mathbf{v})^T$

Importance of $\mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbb{O}$

$\mathbb{T} := -p\mathbb{I} + \mathbb{S} = (\frac{1}{3}\operatorname{tr}\mathbb{T})\mathbb{I} + \big(\mathbb{T} - (\frac{1}{3}\operatorname{tr}\mathbb{T})\mathbb{I}\big)$ $\qquad 2\mathbb{D} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$

$$\boxed{\mathbb{D} = \frac{\alpha(|\mathbb{S}|^2)}{2\nu(|\mathbb{D}|^2)}\,\frac{(|\mathbb{S}| - \tau_*)^+}{(\tau_* + (|\mathbb{S}| - \tau_*)^+)}\mathbb{S} \quad \text{with } \tau_* \geq 0}$$

$x^+ = \max\{x, 0\}$, ν and α are positive functions

If $\tau_* = 0$ the above formula includes as special cases

- Navier-Stokes fluids: $\mathbb{S} = 2\nu_*\mathbb{D}$
- power-law fluids: $\mathbb{S} = 2\nu_*(\beta_* + |\mathbb{D}|^2)^{\frac{r-2}{2}}\mathbb{D}$
- stress power-law fluids: $\mathbb{D} = \frac{1}{2\nu_*}(\gamma_* + |\mathbb{S}|^2)^{\frac{r'-2}{2}}\mathbb{S}$

$r \in (1,\infty)$, $r' := r/(r-1)$, and ν_*, β_*, γ_* are positive constants

Importance of $\mathbb{G}(\mathbb{S}, \mathbb{D}) = \mathbb{O}$

$$\boxed{\mathbb{D} = \frac{\alpha(|\mathbb{S}|^2)}{2\nu(|\mathbb{D}|^2)} \frac{(|\mathbb{S}| - \tau_*)^+}{(\tau_* + (|\mathbb{S}| - \tau_*)^+)} \mathbb{S} \quad \text{with } \tau_* \geq 0}$$

$x^+ = \max\{x, 0\}$, ν and α are positive functions

If $\alpha \equiv 1$ and $\tau_* > 0$ the above formula is equivalent to

$$|\mathbb{S}| \leq \tau_* \Leftrightarrow \mathbb{D} = \mathbf{0} \quad \text{and} \quad |\mathbb{S}| > \tau_* \Leftrightarrow \mathbb{S} = \frac{\tau_* \mathbb{D}}{|\mathbb{D}|} + 2\nu(|\mathbb{D}|^2)\mathbb{D}$$

and includes as special cases

- Bingham fluids (1923): $\nu(s) = \nu_*$, $\alpha(s) = 1$ and $\tau_* > 0$
- Herschel-Bulkley fluids (1926): $\nu(s)$ as for power-law fluids, $\alpha(s) = 1$ and $\tau_* > 0$

Importance of $\mathbb{G}(\mathbb{T},\mathbb{D}) = \mathbb{O}$

NAVIER-STOKES FLUID can not describe several phenomena that have been observed and documented experimentally:

- **shear thinning, shear thickening** - ν depends on $|\mathbb{D}|^2$
- **pressure thickening** - ν depends on p
- **the presence of activation or deactivation criteria** - "jump" singularities
- **the presence of the normal stress differences at simple shear flows**
- stress relaxation
- non-linear creep
- responses of anisotropic fluids, ...

$\mathbb{G}(\mathbb{T},\mathbb{D}) = \mathbb{O}$ has potential to describe four of them - rich structure.

Models connected with names like Ostwald (1925), de Waele (1923), Carreau (1972), Yasuda (1979), Eyring (1958), Cross (1965), Sisko (1958), Matsuhisa and Bird (1965), Glen (1955), Blatter (1995), Barus (1893), Bingham (1922) etc.

Considerations leading to the assumptions on $\mathbb{G}(\cdot,\cdot)$

The quantity $\mathbb{S}\cdot\mathbb{D}$ for $\mathbb{S} = 2\nu_*\mathbb{D} \iff \mathbb{D} = \frac{1}{2\nu_*}\mathbb{S}$

$$\begin{aligned}\mathbb{S}\cdot\mathbb{D} &= 2\nu_*|\mathbb{D}|^2\\ &= \tfrac{1}{2\nu_*}|\mathbb{S}|^2\\ &= \nu_*|\mathbb{D}|^2 + \tfrac{1}{4\nu_*}|\mathbb{S}|^2\\ &= \psi(|\mathbb{D}|) + \psi^*(|\mathbb{S}|) \qquad\qquad \psi^*(s) = \max_{t\in[0,\infty)}(s\cdot t - \psi(t))\end{aligned}$$

Similarly and explicitly for $\mathbb{S} = 2\mu^*|\mathbb{D}|^{r-2}\mathbb{D} \iff \mathbb{D} = [2\mu^*]^{-\frac{1}{r-1}}|\mathbb{S}|^{\frac{2-r}{r-1}}\mathbb{S}$

$$\begin{aligned}\mathbb{S}\cdot\mathbb{D} &= |\mathbb{D}|^r = |\mathbb{S}|^{r/(r-1)}\\ &= \tfrac{\mathbb{S}\cdot\mathbb{D}}{r} + \tfrac{\mathbb{S}\cdot\mathbb{D}}{r'} = \tfrac{|\mathbb{D}|^r}{r} + \tfrac{|\mathbb{S}|^{r'}}{r'} = \psi(|\mathbb{D}|) + \psi^*(|\mathbb{S}|)\end{aligned}$$

Monotone and maximal monotone response

Power-law fluids (similarly also stress power-law fluids and their generalizations)

$$\boxed{\mathbb{S} = 2\mu^* |\mathbb{D}|^{r-2}\mathbb{D} \iff \mathbb{D} = [2\mu^*]^{-\frac{1}{r-1}} |\mathbb{S}|^{\frac{2-r}{r-1}} \mathbb{S}}$$

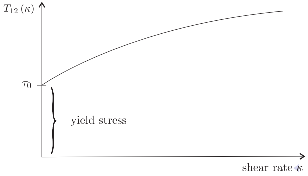
For all $\mathbb{D}$, $\mathbb{E} \in \mathbb{R}^{3\times 3}$

$$\left(\tilde{\mathbb{S}}(\mathbb{D}) - \tilde{\mathbb{S}}(\mathbb{E})\right) \cdot (\mathbb{D} - \mathbb{E}) \geq 0 \qquad \tilde{\mathbb{S}}(\mathbb{B}) := 2\mu^* |\mathbb{B}|^{r-2}\mathbb{B}$$

For all $\mathbb{S}_1$, $\mathbb{S}_2 \in \mathbb{R}^{3\times 3}$

$$(\mathbb{S}_1 - \mathbb{S}_2) \cdot \left(\mathbb{B}(\mathbb{S}_1) - \mathbb{B}(\mathbb{S}_2)\right) \geq 0\,, \qquad \text{where} \quad \mathbb{B}(\mathbb{S}) := 2\mu^* |\mathbb{S}|^{\frac{2-r}{r-1}} \mathbb{S}$$

Bingham and Herschel-Bulkley fluids



Implicit formulation - maximal monotone ψ-graph setting

$$\boxed{(\mathbb{S},\mathbb{D}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbf{0}}$$

Assumptions ($\mathcal{A}$ is a ψ-maximal monotone graph):

(A1) $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$

(A2) **Monotone graph:** For any $(\mathbb{S}_1,\mathbb{D}_1),(\mathbb{S}_2,\mathbb{D}_2) \in \mathcal{A}$

$$(\mathbb{S}_1 - \mathbb{S}_2)\cdot(\mathbb{D}_1 - \mathbb{D}_2) \geq 0$$

(A3) **Maximal graph:** If for some $(\mathbb{S},\mathbb{D})$ there holds

$$(\mathbb{S} - \tilde{\mathbb{S}})\cdot(\mathbb{D} - \tilde{\mathbb{D}}) \geq 0 \qquad \forall\, (\tilde{\mathbb{S}},\tilde{\mathbb{D}}) \in \mathcal{A}$$

then

$$(\mathbb{S},\mathbb{D}) \in \mathcal{A}$$

(A4) **ψ-graph:** There are $c_* \in (0,1]$ and $g > 0$ so that for any $(\mathbb{S},\mathbb{D}) \in \mathcal{A}$

$$\mathbb{S}\cdot\mathbb{D} \geq c_*(\psi(\mathbb{D}) + \psi^*(\mathbb{S})) - g \quad \text{or} \quad \mathbb{S}\cdot\mathbb{D} \geq c_*\left(|\mathbb{D}|^r + |\mathbb{S}|^{r'}\right) - g$$

Definition of weak solution to the Problem with Navier's slip bcs

Definition

We say $(p, \mathbf{v}, \mathbb{S})$ is weak solution to *Problem*

$$p \in L^1(Q)$$

$$\mathbf{v} \in C_{\mathrm{weak}}(0, T; L^2_{\mathbf{n},\mathrm{div}}) \cap L^q(0, T; W^{1,q}_{\mathbf{n},\mathrm{div}}) \text{ with } \mathbb{D}(\mathbf{v}) \in L^{\psi}(Q)$$

$$\mathbb{S} \in L^{\psi^*}(Q)$$

$$\lim_{t \to 0_+} \|\mathbf{v}(t) - \mathbf{v}_0\|_2^2 = 0$$

$$\langle \mathbf{v}', \mathbf{w} \rangle + (\mathbb{S}, \mathbb{D}(\mathbf{w})) - (\mathbf{v} \otimes \mathbf{v}, \mathbb{D}(\mathbf{w})) + \alpha_*(\mathbf{v}_{\boldsymbol{\tau}}, \mathbf{w}_{\boldsymbol{\tau}})_{\partial\Omega} = \langle \mathbf{b}, \mathbf{w} \rangle, +(p, \operatorname{div} \mathbf{w}),$$

$$\text{for all } \mathbf{w} \in W^{1,1}_{\mathbf{n}} \text{ such that } \mathbb{D}(\mathbf{w}) \in L^{\infty}(\Omega)^{d \times d} \text{ and a.a. } t \in (0, T),$$

$$(\mathbb{D}(\mathbf{v}(t, x)), \mathbb{S}(t, x)) \in \mathcal{A} \text{ for a.a. } (t, x) \in Q.$$

Theorem

Theorem

Let $\mathcal{A}$ satisfy the assumptions **(A1)–(A4)** *with ψ fulfilling*

$$\boxed{c_1 s^r - c_2 \le \psi(s) \le c_3 s^{\tilde{r}} + c_4 \quad \text{with } r > \frac{6}{5}}$$

Then for any $\Omega \in \mathcal{C}^{1,1}$ and $T \in (0, \infty)$ and for arbitrary

$$\mathbf{v}_0 \in L^2_{\mathbf{n},\mathrm{div}}, \quad \mathbf{b} \in L^2(0, T; L^2(\Omega)^3) \ \text{ and } \gamma_* \ge 0\,, \tag{1}$$

there exists weak solution to Problem.

Novel tools:
(i) structural assumptions on $\mathbb{G}(\mathbb{T}, \mathbb{D}) = \mathbb{O}$
(ii) convergence criterion
(iii) understanding the interplay between the chosen boundary conditions and global integrability of p
(iv) Lipschitz approximations of Sobolev and Bochner functions

M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda: On Unsteady Flows of Implicitly Constituted Incompressible Fluids, *SIAM J. Math. Anal.*, Vol. 44, No. 4, pp. 2756–2801 (2012)

Methods

- subcritical case
  - Minty's method
  - energy equality - $\mathbf{v}$ is an addmissible test function
- supercritical case
  - generalized Minty's method
  - Lipschitz approximation in Orlicz-Sobolev spaces

Generalized Minty's method - Convergence lemma

Assume that

- $\mathcal{A}$ is a maximal monotone ψ-graph satisfying **(A1)–(A4)**
- $\{\mathbb{S}^n\}_{n=1}^{\infty}$ and $\{\mathbb{D}^n\}_{n=1}^{\infty}$ satisfy for some $Q' \subset Q$

$$
\begin{aligned}
(\mathbb{S}^n, \mathbb{D}^n) &\in \mathcal{A} && \text{for a.a. } (t,x) \in Q', \\
\mathbb{D}^n &\rightharpoonup \mathbb{D} && \text{weakly in } L^{\psi}(Q'), \\
\mathbb{S}^n &\rightharpoonup \mathbb{S} && \text{weakly in } L^{\psi^*}(Q'),
\end{aligned}
$$

$$
\limsup_{n\to\infty} \int_{Q'} \mathbb{S}^n \cdot \mathbb{D}^n \, dx\, dt \le \int_{Q'} \mathbb{S} \cdot \mathbb{D} \, dx\, dt.
$$

Then for almost all $(t,x) \in Q'$ we have

$$
(\mathbb{S}, \mathbb{D}) \in \mathcal{A}
$$

Lemma - *Local* version

Importance of $\mathbf{g}((\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}, \mathbf{v}_{\boldsymbol{\tau}}) = \mathbf{0}$

$$\boxed{b(|\mathbf{v}_{\boldsymbol{\tau}}|)\mathbf{v}_{\boldsymbol{\tau}} = a(|(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}|^2)\frac{(|(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}| - \sigma_*)^+}{(\sigma_* + (|(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}| - \sigma_*)^+)}(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}} \quad \text{with } \sigma_* \geq 0}$$

$x^+ = \max\{x, 0\}$, a is a positive and b nonnegative functions which includes as subcases

- slip condition: $a(s) \equiv 1$, $b(s) = 0$ and $\sigma_* = 0$,
- Navier's slip condition: $a(s) = (1-\lambda)\alpha_* > 0$, $b(s) = \lambda$ with $\lambda \in (0,1)$,
- threshold slip: $a(s) \equiv 1$ and $\sigma_* > 0$

Threshold slip condition can be written equivalently as:

$$|(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}| \leq \sigma_* \Leftrightarrow \mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{and} \quad |(\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}}| > \sigma_* \Leftrightarrow (\mathbb{S}\mathbf{n})_{\boldsymbol{\tau}} = \sigma_* \frac{\mathbf{v}_{\boldsymbol{\tau}}}{|\mathbf{v}_{\boldsymbol{\tau}}|} + b(|\mathbf{v}_{\boldsymbol{\tau}}|)\mathbf{v}_{\boldsymbol{\tau}}$$

- no-slip condition: $a(s) = 0$

No-slip versus Threshold slip

- Homogeneous Dirichlet boundary conditions are considered as the simplest for many PDEs
- In incompressible fluid dynamical problems, it is open if $p \in L^{Q_T}$ for no-slip boundary conditions for problems where the viscosity is not constant
- The difficulty is due to incompatibility of the no-slip bcs with Helmholtz decomposition. Indeed for $\varphi \in W^{1,r}_{\mathbf{n}}$

$$\begin{aligned}\langle \tfrac{\partial \mathbf{v}}{\partial t}, \varphi \rangle &= \langle \tfrac{\partial \mathbf{v}}{\partial t}, \varphi^{\mathrm{div}} + \nabla h \rangle \\ &= \langle \tfrac{\partial \mathbf{v}}{\partial t}, \varphi^{\mathrm{div}} \rangle = \texttt{weak form of balance of linear momentum}\end{aligned}$$

Implicit formulation - maximal monotone q-graph setting

$$\boxed{(\mathbf{s},\mathbf{v}_\tau)\in\mathcal{B} \iff \mathbf{g}(\mathbf{s},\mathbf{v}_{\boldsymbol\tau})=\mathbf{0}}$$

(B1) *$\mathcal{B}$ comes through the origin.* $(\mathbf{0},\mathbf{0})\in\mathcal{B}$.

(B2) *$\mathcal{B}$ is a monotone graph.*

$$(\mathbf{s}_1-\mathbf{s}_2)\cdot(\mathbf{v}_{\boldsymbol\tau}^1-\mathbf{v}_{\boldsymbol\tau}^2)\geq 0 \qquad \text{for all } (\mathbf{s}_1,\mathbf{v}{\boldsymbol\tau}^1),(\mathbf{s}_2,\mathbf{v}_{\boldsymbol\tau}^2)\in\mathcal{B}.$$

(B3) *$\mathcal{B}$ is a maximal monotone graph.* Let $(\mathbf{s},\mathbf{u})\in\mathcal{R}^3\times\mathcal{R}^3$ be given.

$$\text{If}\quad (\bar{\mathbf{s}}-\mathbf{s})\cdot(\bar{\mathbf{v}}_{\boldsymbol\tau}-\mathbf{u})\geq 0 \quad \text{for all } (\bar{\mathbf{s}},\bar{\mathbf{v}}_{\boldsymbol\tau})\in\mathcal{B} \quad \text{then } (\mathbf{s},\mathbf{u})\in\mathcal{B}.$$

(B4) *$\mathcal{B}$ is a q-graph.* For any $q\in(1,\infty)$ fixed there are $d_*>0$ and $n_*\geq 0$ such that

$$\mathbf{s}\cdot\mathbf{v}_{\boldsymbol\tau}\geq -n_*+d_*(|\mathbf{v}_{\boldsymbol\tau}|^q+|\mathbf{s}|^{q/(q-1)}) \qquad \text{for all } (\mathbf{s},\mathbf{v}_{\boldsymbol\tau})\in\mathcal{B}\,.$$

No-slip boundary condition is excluded by **(B4)**.

Concluding Remarks

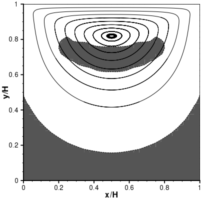
Implicit constitutive theory

- robust framework (the same number of quantities capable of describing large class of material responses) that provides a new look at the fluids with activation or deactivation criteria in the bulk and/or on the boundary
- far reaching consequences to the theoretical foundation of continuum mechanics and thermodynamics (introduced as an object for a systematic studies by KR Rajagopal in 2003, and develop further in many publications)
- threshold slip is the way how to overcome the troubles connected with the analysis of unsteady flows subject to homogeneous Dirichlet boundary conditions (no-slip)
- for imlicitly constituted fluids characterized by **(A1)**-**(A4)** and $r > 6/5$, we define the solution and show its large data existence - object to be studied numerically and computationally. It provides a suitable framework to link (new approaches in) modeling with analysis of matrix computation to develop efficient numerical methods
- new options how to numerically discretize the problems - some give interesting results (second order vs. first order PDEs)

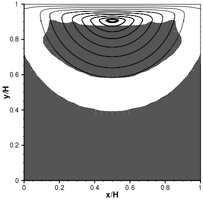
Lid driven cavity with Bingham fluid (J. Hron, J. Málek, J. Stebel, K. Touška)

- Unknowns $(\mathbf{v}, p, \mathbb{S})$:

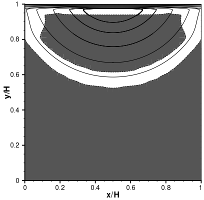
$$
\begin{aligned}
-\operatorname{div} \mathbb{S} &= -\nabla p + \mathbf{b} \\
\mathbb{G}(\mathbb{S}, \mathbb{D}) &= \mathbb{O} \\
\mathbb{D}(\mathbf{v}) &= \mathbb{D} \qquad \text{improves convergence for larger } \tau_*
\end{aligned}
$$



$\tau^* = 5$



$\tau^* = 50$



$\tau^* = 500$

D. Vola, L. Boscardin, J.C. Latché: Laminar unsteady flows of Bingham fluids: a numerical strategy and some benchmark results, 2003.

Concluding Remarks

• Implicitly constituted material models: from theory through model reduction to efficient numerical methods (5 year ERC-CZ project MORE financed by the Ministry of Education, Youth and Sports)

- Team members: Z. Strakoš, E. Feireisl, E. Süli (Oxford), M. Bulíček, J. Hron, V. Průša, O. Souček and M. Vohralík (INRIA)
- Advisory board members: M. Benzi (Atlanta), K.R. Rajagopal (Texas A&M University), R. Rannacher (Heidelberg), G. Seregin (Oxford, St. Petersburg)
- two postdoc positions since February 2013 taken by 5 postdocs in 2013, Ph.D. students: J. Papež, M. Řehoř, J. Žabenský

Workshop: Implicitly constituted materials: Modeling, Analysis and Computation (Chateau Liblice, November 24 - 27, 2013)

• 13th International School on Mathematical Theory in Fluid Mechanics (Kácov, Czech republic, May 24 - 31, 2013)